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On Well Posed Generalized Best Approximation Problems[1]

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Let C be a closed bounded convex subset of X with 0 being an interior point of C and p_C be the Minkowski functional with respect to C. Let G be a nonempty closed, boundedly relatively weakly compact subset of a Banach space X. For a point $x \in X$, we say the minimization problem $\min_C(x, G)$ is well posed if there exists a unique point $\bar{z}$ such that $p_C(\bar{z}-x)=\lambda_C(x, G)$ and every sequence $\{z_n\} \subset G$ satisfying $\lim_{n\to\infty} p_C(z_n-x)=\lambda_C(x, G)$ converges strongly to the point $\bar{z}$, where $\lambda_C(x, G)=\inf_{z\in G} p_C(z-x)$. Under the assumption that C is both strictly convex and Kadec, we prove that the set $X_o(G)$ of all $x \in X$ such that the problem $\min_C(x, G)$ is well posed is a residual subset of X extending the results in the case that the modulus of convexity of C is strictly positive due to Blasi and Myjak. In addition, we also prove these conditions are necessary. © 2000 Academic Press

1. INTRODUCTION

Let X be a real Banach space of dimension at least 2 and X^* be the dual of X. For a nonempty subset, $A \subset X$, as usual, by int A and ∂A we mean the interior of A and the boundary of A, respectively, while $[x, y]$ stands for the closed interval with end points x and y. We use $B(x, r)$ to denote the closed ball in X with center x and radius r. In particular, we put $B=B(0, 1)$.

Throughout this paper C will denote a closed bounded convex subset of X with $0 \in \text{int}\, C$. Recall that the functional of Minkowski $p_C: X \mapsto R$ with respect to the set C is defined by

$$p_C(x)=\inf\{\alpha>0 : x \in \alpha C\}. \tag{1.1}$$

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For a closed subset G of X and $x \in X$ put

$$\lambda_C(x, G) = \inf_{z \in G} p_C(z - x). \tag{1.2}$$

Given a nonempty closed subset G of X and $x \in X$, Blasi and Myjak [3] considered the minimization problem, denoted by $\min_C(x, G)$, which consists in fining points $\bar{z}$ such that $p_C(\bar{z} - x) = \lambda_C(x, G)$. According to [3], any such point $\bar{z}$ is called a solution of the minimization problem $\min_C(x, G)$ and any sequence $\{z_n\} \subset G$ satisfying $\lim_{n \to \infty} p_C(z_n - x) = \lambda_C(x, G)$ is called a minimizing sequence of the minimization problem $\min_C(x, G)$. The minimization problem $\min_C(x, G)$ is said to be well posed if it has a unique solution, say z_0, and every minimizing sequence converges strongly to z_0.

Let $\delta_C\colon [0, 2] \mapsto [0, +\infty)$ be the modulus of convexity of C, i.e.,

$$\delta_C(\varepsilon) = \inf \left\{ 1 - p_C\left(\frac{x + y}{2}\right) : x, y \in C \text{ and } p_C(x - y) \geqslant \varepsilon \right\}. \tag{1.3}$$

Under the assumption that $\delta_C(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$, it was proved in [3] that, for every nonempty closed subset G of X, the set $X_o(G)$ of all $x \in X$ such that the problem $\min_C(x, G)$ is well posed is a residual subset of X.

In the present paper, using a completely different approach, which was developed by Lau [11] and Borwein and Fitzpatrick [1], we prove that if C is both strictly convex and Kadec, then the set $X_o(G)$ of all $x \in X$ such that the problem $\min_C(x, G)$ is well posed is a residual subset of X provided that G is a closed, bounded relatively weakly compact, nonempty subset of X. We extend the result due to Blasi and Myjak [3]. In addition, we also show these conditions made on C is necessary for $X_o(G)$ to be residual for every closed subset G of X. Further results in the same spirit can be founded in [1–5, 7, 8, 11–13, 16].

2. PRELIMINARIES

For the reader's convenience we first recall some well known properties of the Minkowski functional which follow immediately from the definition.

Proposition 2.1. *For every $x, y \in X$, we have*

(i) $p_C(x) \geqslant 0$ *and* $p_C(x) = 0$ *iff* $x = 0$;

(ii) $p_C(x + y) \leqslant p_C(x) + p_C(y)$;

(iii) $-p_C(y-x) \leqslant p_C(x) - p_C(y) \leqslant p_C(x-y)$;

(iv) $p_C(\lambda x) = \lambda p_C(x)$, *if* $\lambda \geqslant 0$;

(v) $p_C(-x) = p_{-C}(x)$;

(vi) $p_C(x) = 1$ *iff* $x \in \partial C$;

(vii) $p_C(x) < 1$ *iff* $x \in \operatorname{int} C$;

(viii) $\mu \|x\| \leqslant p_C(x) \leqslant \nu \|x\|$,

where and in the following

$$\mu = \inf_{x \in \partial B} p_C(x) \qquad \textit{and} \qquad \nu = \sup_{x \in \partial B} p_C(x).$$

DEFINITION 2.1. C is called strictly convex if $\partial C = \operatorname{ext} C$, the set of all extreme points of C.

From the definition, it follows that C is strictly convex if and only if for any $x, y \in \partial C$, $p_C(x+y) = p_C(x) + p_C(y)$ implies $x = y$.

DEFINITION 2.2. C is called (sequentially) Kadec if any sequence $\{x_n\} \subset \partial C$ with $x_n \to x_0 \in \partial C$ weakly converges strongly to x_0.

PROPOSITION 2.2. *Define*

$$q_C(x^*) = \sup_{x \in C} \langle x^*, x \rangle$$

for every $x^* \in X^*$. *Then*

(i) $q_C(x^* + y^*) \leqslant q_C(x^*) + q_C(y^*)$ *for every* $x^*, y^* \in X^*$;

(ii) $q_C(\lambda x^*) = \lambda q_C(x^*)$ *for all* $\lambda \geqslant 0$ *and* $x^* \in X^*$;

(iii) $p_C(x) = \max\{\langle x^*, x \rangle : x^* \in X^*, q_C(x^*) \leqslant 1\}$.

PROPOSITION 2.3. *Suppose that* $\delta_C(\varepsilon) > 0$ *for each* $\varepsilon \in (0, 2]$. *Then*

(i) *C is strictly convex*;

(ii) *C is Kadec*;

(iii) *X is reflexive.*

Proof. (i) The strict convexity results from Proposition 2.4 of [3].

(ii) Let $\{x_n\} \subset \partial C$ and $x_0 \in \partial C$ satisfying $x_n \to x_0$ weakly. Taking $x_0^* \in X^*$ with $\langle x_0^*, x_0 \rangle = p_C(x_0^*) = 1$, we have that

$$2 \geqslant \limsup_{n \to \infty} p_C(x_n + x_0) \geqslant \lim_{n \to \infty} \langle x_0^*, x_n + x_0 \rangle = 2,$$

and so

$$\lim_{n \to \infty} \delta_C(p_C(x_n + x_0)) = 0.$$

Now the fact that $\delta_C(\varepsilon) > 0$ for $\forall \varepsilon \in (0, 2]$ shows $\lim_{n \to \infty} p_C(x_n - x_0) = 0$ and consequently, $\lim_{n \to \infty} \|x_n - x_0\| = 0$, i.e., C is Kadec.

(iii) By James' theorem [9], it suffices to prove that for each $x^* \in X^*$ with $q_C(x^*) = 1$ there exists $x_0 \in C$ such that $\langle x^*, x_0 \rangle = 1$. For the end, let $\{x_n\} \subset C$ satisfying

$$\lim_{n \to \infty} \langle x^*, x_n \rangle = 1.$$

Then

$$2 \geqslant \limsup_{n, m \to \infty} p_C(x_n + x_m) \geqslant \lim_{n, m \to \infty} \langle x^*, x_n + x_m \rangle = 2,$$

and

$$\lim_{n, m \to \infty} \delta_C(p_C(x_n + x_m)) = 0.$$

This, with the fact that $\delta_C(\varepsilon) > 0$ for $\forall \varepsilon \in (0, 2]$, implies that $\lim_{n \to \infty} p_C(x_n - x_m) = 0$ and consequently, $\lim_{n \to \infty} \|x_n - x_m\| = 0$ so that $\lim_{n \to \infty} \|x_n - x_0\| = 0$ and $\langle x^*, x_0 \rangle = 1$ for some $x_0 \in C$. The proof is complete. ∎

Remark. Obviously, C is both strictly convex and Kadec if and only if so is $-C$.

Finally, we also need the concept of Frechet differentiability and a result on the Frechet differentiability of Lipschitz functions due to [15].

DEFINITION 2.4. Let D be an open subset of X. A real-valued function f on D is said to be Frechet differentiable at $x \in D$ if there exists an $x^* \in X^*$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} = 0.$$

x^* is called the Frechet differential at x which is denoted by $\mathrm{D}f(x)$.

PROPOSITION 2.3. *Let f be a locally Lipschitz continuous function on an open set D of a Banach space with equivalent Frechet differentiable norm (in particular, X reflexive will do). Then f is Frechet differentiable on a dense subset of D.*

3. WELL POSED GENERALIZED BEST APPROXIMATION

Let G be a closed subset of X and $x \in X$. Set

$$\lambda_C^+(x, G) = \inf_{z \in G} p_C(x - z). \tag{3.1}$$

From Proposition 2.1(v), it follows that the problem $\min_C(x, G)$ is well posed if and only if the minimization problem $\min^+_{-C}(x, G)$, which consists in finding a point $\bar{z} \in G$ satisfying $p_{-C}(x - \bar{z}) = \lambda^+_{-C}(x, G)$, is well posed, where the concepts of the solution, the minimizing sequence and the well posedness are defined similarly.

For notational convenience, let $d_G(x) = \lambda_C^+(x, G)$. Then, we have

PROPOSITION 3.1. (i) $d_G(x) - d_G(y) \leqslant p_C(x - y)$, $\forall x, y \in X$;

(ii) $|d_G(x) - d_G(y)| \leqslant v \, \|x - y\|$, $\forall x, y \in X$.

Let

$$L_n(G) = \left\{ x \in X \backslash G : \begin{array}{l} \inf\{\langle x^*, x - z\rangle : z \in G \cap B_C(x, d_G(x) + \delta)\} \\ > (1 - 2^{-n}) \, d_G(x), \text{ for some } \delta > 0, x^* \in X^* \\ \text{with } q_C(x^*) = 1 \end{array} \right\},$$

where $B_C(x, r) = \{ y \in X : p_C(x - y) \leqslant r\}$.

Also let

$$L(G) = \bigcap_n L_n(G)$$

and let

$$\Omega(G) = \left\{ x \in X \backslash G : \begin{array}{l} \text{there exists } x^* \in X^* \text{ with } q_C(x^*) = 1, \text{ such that for} \\ \text{each } \varepsilon > 0 \text{ there is } \delta > 0 \text{ so that } \inf\{\langle x^*, x - z\rangle : \\ z \in G \cap B_C(x, d_G(x) + \delta)\} > (1 - \varepsilon) \, d_G(x) \end{array} \right\}.$$

Obviously, $\Omega(G) \subset L(G)$.

LEMMA 3.1. *$L(G)$ is a G_δ-subset of X.*

Proof. To show that $L(G)$ is a G_δ-subset of X, we only need prove that $L_n(G)$ is open for each n. Let $x \in L_n(G)$. Then there exist $x^* \in X^*$ with $q_C(x^*) = 1$ and $\delta > 0$ such that

$$\beta = \inf\{\langle x^*, x - z\rangle : z \in G \cap B_C(x, d_G(x) + \delta)\} - (1 - 2^{-n}) \, d_G(x) > 0.$$

Let $\lambda > 0$ be such that $\lambda < \min\{\frac{\delta}{2}, \frac{\beta}{2}\}$ and fix y with $\|x - y\| < \lambda/v$. For $\delta^* = \delta - 2\lambda$, from Proposition 3.1(i), we have

$$H = G \cap B_C(y, d_G(y) + \delta^*) \subset G \cap B_C(x, d_G(x) + \delta).$$

Thus if $z \in H$,

$$\langle x^*, x - z \rangle \geqslant \beta + (1 - 2^{-n})\, d_G(x)$$

and

$$\begin{aligned}
&\langle x^*, y - z \rangle \\
&\quad = \langle x^*, y - x \rangle + \langle x^*, x - z \rangle \\
&\quad \geqslant \beta + (1 - 2^{-n})\, d_G(y) - p_C(x - y) - (1 - 2^{-n})(d_G(y) - d_G(x)) \\
&\quad \geqslant \beta + (1 - 2^{-n})\, d_G(y) - p_C(x - y) - p_C(y - x) \\
&\quad \geqslant \beta + (1 - 2^{-n})\, d_G(y) - 2v\, \|x - y\| \\
&\quad \geqslant (1 - 2^{-n})\, d_G(y) + \beta - 2\lambda.
\end{aligned}$$

Then

$$\inf\{\langle x^*, y - z \rangle : z \in H\} > (1 - 2^{-n})\, d_G(y)$$

and $y \in L_n(G)$ for all $y \in X$ with $v\, \|x - y\| < \lambda$, which implies that $L_n(G)$ is open in X. ▮

The following factorization theorem due to Davis, Figiel, Johnson and Pelczynski [6] plays a key role in the proof of the density of $\Omega(G)$.

THEOREM DFJP. *Let K be a weakly compact subset of a Banach space Y with $Y = \overline{\text{span}\, K}$. Then there exists a reflexive Banach space R and a one to one continuous linear mapping T: $R \mapsto Y$ such that $T(B) \supset K$.*

LEMMA 3.2. *If G is a closed, boundedly relatively weakly compact, non-empty subset of X, then $\Omega(G)$ is dense in $X \backslash G$.*

Proof. Let $x_0 \in X \backslash G$ and $d_G(x_0) > \varepsilon > 0$. Let

$$K = \text{weak} - \text{cl}\{(B(0, N) \cap G) \cup \{x_0\}\},$$

where $N = \|x\| + \frac{3d_G(x)}{\mu}$. Then K is weakly compact and if $Y = \overline{\text{span}\, K}$, we can apply Theorem DFJP to obtain a reflexive Banach space R and a one to one continuous linear mapping T: $R \mapsto Y$ such that $T(B) \supset K$. Define

$$f_G(u) = d_G(Tu), \qquad \text{for each} \quad u \in R.$$

Then f_G is a Lipschitz function on R and so by Proposition 2.4 f_G is Frechet differentiable on a dense subset of R. Thus there exists differentiable point $v \in R$ of f_G with $\mathrm{D}f_G(v) = v^*$ such that $y := Tv \in B(x_0, \varepsilon)$. This means that

$$\lim_{h \to 0} \frac{d_G(T(v+h)) - d_G(Tv) - \langle v^*, h \rangle}{\|h\|} = 0$$

and hence

$$\lim_{h \to 0} \frac{d_G(y+Th) - d_G(y) - \langle v^*, h \rangle}{p_C(h)} = 0.$$

Substituting tu for h in the previous expression and using Proposition 3.1 we have

$$\langle v^*, u \rangle \leqslant p_C(Tu).$$

This shows $v^* = T^* y^*$ for some $y^* \in Y^*$. Furthermore, $\langle y^*, Tu \rangle \leqslant p_C(Tu)$ for all $u \in R$ so that $q_C(y^*) \leqslant 1$ since T has dense range. By Hahn–Banach theorem we may extend y^* to x^* with $q_C(x^*) \leqslant 1$. Now let $\{z_n\}$ be a minimizing sequence in G for y. Then for each $1 \geqslant t > 0$,

$$\begin{aligned} d_G(y + t(z_n - y)) - d_G(y) &\leqslant p_C(y + t(z_n - y) - z_n) - d_G(y) \\ &= (1-t)\, p_C(y - z_n) - d_G(y) \\ &= -t p_C(y - z_n) + [p_C(y - z_n) - d_G(y)]. \end{aligned}$$

Let $t_n = 2^{-n} + [p_C(y - z_n) - d_G(y)]^{1/2}$. Observe that

$$\lim_{n \to \infty} \frac{d_G(y + t_n(z_n - y)) - d_G(y) - t_n \langle x^*, z_n - y \rangle}{t_n} = 0.$$

We have that

$$\liminf_{n \to \infty} [-p_C(y - z_n) + \langle x^*, y - z_n \rangle] \geqslant 0$$

and

$$d_G(y) = \lim_{n \to \infty} p_C(y - z_n) \leqslant \liminf_{n \to \infty} \langle x^*, y - z_n \rangle,$$

which again shows $q_C(x^*) \geqslant 1$. Thus $q_C(x^*) = 1$ and

$$d_G(y) = \lim_{n \to \infty} \langle x^*, y - z_n \rangle.$$

This implies $y \in \Omega(G)$ and proves the Lemma. ∎

THEOREM 3.1. *Suppose that C is both strictly convex and Kadec. Let G be a closed, boundedly relatively weakly compact, non-empty subset of X, then the set $X_0^+(G)$ of all $x \in X$ such that the problem $\min_C^+(x, G)$ is well posed is a residual subset of X.*

Proof. From Lemma 3.1 and 3.2 it suffices to prove that for each $x \in L(G)$ the problem $\min_C^+(x, G)$ is well posed. Now let $x \in L(G)$. We first show that $\min_C^+(x, G)$ has a unique solution. Suppose that $\min_C^+(x, G)$ has two solutions z_0, z_1. Since $x \in L(G)$, for each n, there exists $x_n^* \in X^*$, $q_C(x_n^*) = 1$ satisfying

$$\langle x_n^*, x - z_i \rangle > (1 - 2^{-n})\, d_G(x), \qquad i = 0, 1$$

so that

$$p_C(x - z_0 + x - z_1) \geqslant \limsup_{n \to \infty} \langle x_n^*, x - z_0 + x - z_1 \rangle \geqslant 2\, d_G(x).$$

Thus, using the strict convexity of C we have $z_0 = z_1$, proving the uniqueness.

Next, let $z_n \in G$ be any minimizing sequence for x. Then $x \in L_m(G)$ for any $m = 1, 2, \ldots$. It follows that there exist $\delta_m > 0$, $x_m^* \in X^*$ with $q_C(x_m^*) = 1$ such that

$$\inf\{\langle x_m^*, x - z \rangle : z \in G \cap B_C(x, d_G(x) + \delta_m)\} > (1 - 2^{-m})\, d_G(x).$$

With no loss of generality, we may assume that $z_n \to z_0$ weakly as $n \to \infty$ for some $z_0 \in X$, since G is boundedly relatively weakly compact. Then we have that

$$p_C(x - z_0) \leqslant \liminf_{n \to \infty} p_C(x - z_n) = d_G(x).$$

We also assume that $\delta_n \leqslant \delta_m$ if $m < n$ and so $z_n \in G \cap B_C(x, d_G(x) + \delta_m)$ for all $n > m$. Thus,

$$\langle x_m^*, x - z_n \rangle > (1 - 2^{-m})\, d_G(x), \qquad \forall n > m$$

so that

$$\langle x_m^*, x - z_0 \rangle > (1 - 2^{-m})\, d_G(x), \qquad \forall m.$$

Hence we have

$$p_C(x - z_0) \geqslant \limsup_{m \to \infty} \langle x_m^*, x - z_0 \rangle \geqslant d_G(x).$$

This shows that $p_C(x - z_0) = d_G(x)$. Now the fact that C is Kadec implies that $\lim_{n \to \infty} \|z_n - z_0\| = 0$ and $z_0 \in G$. Clearly, z_0 is a solution of the minimization problem $\min_C^+(x, G)$. In fact we have proved the fact that any minimizing sequence $\{z_n\} \subset G$ for x has a subsequence converging strongly to a solution z_0 of the problem $\min_C^+(x, G)$. Thus the uniqueness of the solution shows that z_n converges to z_0 strongly and completes the proof. ▮

THEOREM 3.1'. *Suppose that C is both strictly convex and Kadec. Let G be a closed, boundedly relatively weakly compact, non-empty subset of X, then the set $X_o(G)$ of all $x \in X$ such that the problem* $\min_C(x, G)$ *is well posed is a residual subset of X.*

COROLLARY 3.1. *Let X be reflexive. Suppose that C is both strictly convex and Kadec. Then for any closed, non-empty subset G of X, the set $X_o(G)$ of all $x \in X$ such that the problem* $\min_C(x, G)$ *is well posed is a residual subset of X.*

COROLLARY 3.2 [3]. *Suppose that $\delta_C(\varepsilon) > 0$ for $\forall \varepsilon \in (0, 2]$. Then for any closed, non-empty subset G of X, the set $X_o(G)$ of all $x \in X$ such that the problem* $\min_C(x, G)$ *is well posed is a residual subset of X.*

THEOREM 3.2. *Suppose that either X is a Banach space which is not reflexive or C is not Kadec. Then there exists a closed bounded non-empty set G in X and an open non-empty subset U of $X \backslash G$ such that for each $x \in U$ the problem* $\min_C^+(x, G)$ *has no solution.*

Proof. *Case* 1. X is not reflexive. By James' theorem [9] there is $x^* \in X^*$ with $1 = q_C(x^*) > \langle x^*, y \rangle$ for each $y \in \partial C$. Let

$$G = B \cap \{z \in X : \langle x^*, z \rangle = 0\}$$

and

$$U = \operatorname{int} B\left(0, \frac{\mu}{2\nu + \mu}\right) \cap \{x \in X : \langle x^*, x \rangle > 0\}.$$

For $x \in U$, we will show that $d_G(x) = \langle x^*, x \rangle$ but the problem $\min_C^+(x, G)$ has no solution. Let $y_n \in C$ such that $\lim \langle x^*, y_n \rangle = 1$ so that we may assume that $\langle x^*, y_n \rangle > 1/2$ for all n. If $x \in U$, set

$$z_n = x - \frac{\langle x^*, x \rangle}{\langle x^*, y_n \rangle} y_n.$$

Then

$$\|z_n\| \leqslant \|x\| + 2p_C(x) \|y_n\| \leqslant \left(1 + 2\frac{v}{\mu}\right) \|x\| < 1.$$

Thus $z_n \in G$ and

$$\begin{aligned} d_G(x) &\leqslant \liminf_n p_C(x - z_n) \\ &= \liminf_n p_C\left(\frac{\langle x^*, x \rangle}{\langle x^*, y_n \rangle} y_n\right) \\ &\leqslant \liminf_n \frac{\langle x^*, x \rangle}{\langle x^*, y_n \rangle} \\ &= \langle x^*, x \rangle. \end{aligned}$$

Suppose a point $z \in G$ such that $d_G(x) = \langle x^*, x \rangle = p_C(x - z)$. Then $\langle x^*, y \rangle = 1$ for $y = \frac{x - z}{p_C(x - z)}$ which is a contradiction.

Case 2. C is not Kadec. By the definition, there exist a sequence $\{y_n\} \subset \partial C$ and a point $y \in \partial C$ such that $y_n \to y$ weakly and $\inf_{n \neq m} \|y_n - y_m\| > \delta$ for some $\delta > 0$. Let $x^* \in X^*$ with $1 = q_C(x^*) = \langle x^*, y \rangle$. Then $\lim_n \langle x^*, y_n \rangle = 1$. With no loss of generality, we may assume that $\langle x^*, y_n \rangle > 1 - 2^{-2(n+1)} > 1/2$ for all n. Set $z_n = (1 + 2^{-n}) y_n$ and define

$$G = \bigcup_n M_n$$

where $M_n = -z_n + B(0, \delta/3) \cap \{z \in X : \langle x^*, z \rangle = 0\}$. Then G is our desired set. First, G is norm closed. In fact, if $n \neq m$ and $u \in M_n$, $w \in M_m$ we have

$$\begin{aligned} \|w - u\| &\geqslant \|y_n - y_m\| - \|y_m - z_m\| - \|y_n - z_n\| - \|z_m + w\| - \|z_n + u\| \\ &\geqslant \delta - 2^{-m} - 2^{-n} - \delta/3 - \delta/3 > \delta/9 \end{aligned}$$

for sufficiently large m, n. Since each M_n is closed, G is closed. Next let $U = \operatorname{int} B(0, \frac{\mu\delta}{3(\mu+2\nu)})$. For $x \in U$ we will show that $d_G(x) = 1 + \langle x^*, x\rangle$ but the problem $\min_C^+(x, G)$ has no solution. For $x \in U$, set

$$w_n = x - z_n - \frac{\langle x^*, x\rangle}{\langle x^*, y_n\rangle} y_n.$$

Then

$$\|w_n + z_n\| \leqslant \|x\| + 2p_C(x)\, \|y_n\| \leqslant \left(1 + 2\frac{\nu}{\mu}\right) \|x\| < \frac{\delta}{3}$$

while $\langle x^*, w_n + z_n\rangle = 0$. Thus $w_n \in M_n$ and

$$\begin{aligned} d_G(x) &\leqslant \liminf_n p_C(x - w_n) \\ &\leqslant \liminf_n p_C\left(z_n + \frac{\langle x^*, x\rangle}{\langle x^*, y_n\rangle} y_n\right) \\ &\leqslant \liminf_n \left(1 + 2^{-n} + \frac{\langle x^*, x\rangle}{\langle x^*, y_n\rangle}\right) \\ &= 1 + \langle x^*, x\rangle. \end{aligned}$$

Now if $z \in G$ then $z \in M_n$ for some n and

$$\begin{aligned} \langle x^*, z\rangle = \langle x^*, -z_n\rangle &= -(1 + 2^{-n})\langle x^*, y_n\rangle \\ &\leqslant -(1 + 2^{-n})(1 - 2^{-2(n+1)}) < -1. \end{aligned}$$

Therefore

$$p_C(x - z) \geqslant \langle x^*, x\rangle - \langle x^*, z\rangle > 1 + \langle x^*, x\rangle$$

and $d_G(x) = 1 + \langle x^*, x\rangle$ but the problem $\min_C^+(x, G)$ has no solution for $x \in U$. The proof is complete. ∎

The following theorem is a generalization of the result on the characterization of strongly convex Banach spaces, which is due to Konjagin [10] and Borwein and Fitzpatrick [1].

THEOREM 3.3. *The following statements are equivalent*

(1) *X is reflexive and C is both strictly convex and Kadec.*

(2) *The function q_C on X^* is Frechet differentiable.*

(3) *For any closed subset G of X, the set $X_o(G)$ is a dense G_δ subset of $X \backslash G$.*

(4) *For any closed subset G of X, the set $X_o(G)$ is a dense subset of $X \backslash G$.*

Proof. $(1) \Rightarrow (3)$ results from Corollary 3.1, while $(3) \Rightarrow (4)$ is trivial.

$(4) \Rightarrow (1)$ by Theorem 3.2 it suffices to prove that C is strictly convex. Suppose on the contrary that there exists two distinct elements, say a, $b \in \partial C$, such that $p_C(a+b) = p_C(a) + p_C(b)$. Take $x^* \in X^*$ with $q(x^*) = 1$ and $\langle x^*, a+b \rangle = 2$, so that $\langle x^*, a \rangle = \langle x^*, b \rangle = 1$. Let

$$G = \{x \in X : \langle x^*, x \rangle = 0\}.$$

Then for any $x \in X$ with $\langle x^*, x \rangle > 0$ there are always multiple solutions to the problem $\min_C^+(x, G)$. Indeed, $d_C(x) = \langle x^*, x \rangle$ and a, b are two solutions to the problem $\min_C^+(x, G)$.

Now let us prove the equivalence of (1) and (2). For this end, let

$$C^{**} = \{x^{**} \in X^{**} : \langle x^*, x^{**} \rangle \leqslant q_C(x^*), \forall x^* \in X\}.$$

Then $C \subset C^{**}$ and $q_C(x^*) = \sup_{x^{**} \in C^{**}} \langle x^*, x^{**} \rangle$. Note that q_C is a Minkowski gauge on X^*, i.e., a nonnegative continuous sublinear functional. Thus from Proposition 5.11 of [14] we have

PROPOSITION 3.2. *q_C is Frechet differentiable at x^* with $\mathrm{D}q_C(x^*) = x^{**}$ if and only if $x^{**} \in C^{**}$ is weakly* strongly exposed by x^* in the sense that for any $\{x_n^{**}\} \subset C^{**}$*

$$\langle x^*, x_n^{**} \rangle \to q_C(x^*) \qquad \textit{implies} \quad \|x_n^{**} - x^{**}\| \to 0.$$

$(1) \Rightarrow (2)$ Observe that $C^{**} = C$ as X^* is reflexive. Thus for any $\{x_n\} \subset C$ with $\langle x^*, x_n \rangle \to q_C(x^*)$, with no loss of generality, assume that $x_n \to x$ weakly for some $x \in C$ so that $\langle x^*, x \rangle = q_C(x^*)$. This implies that $p_C(x_n) \to p_C(x)$. It follows from the fact that C is both strictly convex and Kadec that $\|x_n - x\| \to 0$. Hence $x \in C$ is weakly* strongly exposed by x^* and q_C is Frechet differentiable at x^* with $\mathrm{D}q_C(x^*) = x$.

$(2) \Rightarrow (1)$ For any two x, $y \in \partial C$ with $p_C(x+y) = p_C(x) + p_C(y)$, let $x^* \in X^*$ satisfying $q_C(x^*) = 1$ and $\langle x^*, x+y \rangle = 1$. Then $\langle x^*, x \rangle = \langle x^*, y \rangle = 1$ which, by Lemma 5.10 of [14], implies that $x = y = \mathrm{D}q_C(x^*)$ since q_C is Frechet differentiable at x^*, proving the strict convexity of C. As to the reflexivity, by James' theorem [9], it suffices to show for each $x^* \in X^*$ with $q_C(x^*) = 1$ there exists $x \in C$ such that $\langle x^*, x \rangle = 1$. Let $\{x_n\} \subset C$ such that $\langle x^*, x_n \rangle \to 1$ and let $x^{**} = \mathrm{D}q_C(x^*)$. Then we have

$\|x_n - x^{**}\| \to 0$ from Proposition 3.2 so that $x^{**} \in C$ and $\langle x^*, x \rangle = 1$, completing the proof of the reflexivity. Finally, for the Kadec property of C, let $\{x_n\} \subset \partial C$ satisfying $x_n \to x_0$ weakly for some $x_0 \in \partial C$ and let $x_0^* \in X^*$ with $q_C(x^*) = 1$ and $\langle x_0^*, x_0 \rangle = 1$. Then $x_0 = \mathrm{D}q_C(x^*)$ and $\langle x^*, x_n \rangle \to 1$. Using Proposition 3.2 again, we have $\|x_n - x_0\| \to 0$. The proof is complete. ▮

Remark. Theorem 3.3 extends the results due to Konjagin [10] and Borwein and Fitzpatrick [1].

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